# 2.1 Limits

# Informal Definition

If the values of a function $y = f(x)$ get closer and closer to some number $L$ as values of $x$ get closer and closer to $a$, then we say that $L$ is the limit of $\boldsymbol{f(x)}$ as $x$ approaches $a$. We write

$$\lim_{x \to a} f(x) = \mathrm{L}$$

# Example:

What happens to the y values of the function $f(x) = x^2 - 1$ as x gets closer and closer to 3?

| x | 2.9 | 2.99 | 2.999 | 3 | 3.001 | 3.01 | 3.1 |
|---|---|---|---|---|---|---|---|
| f(x) | 7.41 | 7.9401 | 7.994001 | ? | 8.006001 | 8.0601 | 8.61 |

$$f(2.9) = (2.9)^2 - 1 = 7.41$$
$$f(2.99) = (2.99)^2 - 1 = 7.9401$$
$$\ldots \mathit{Complete\ the\ rest\ of\ the\ table!}$$

# One sided limits

We can break this into the left and ride sides:

1. **Left-Hand Limits:** values of $x$ that are smaller than some number $a$

$$\lim_{x \to a^-} f(x) = K$$

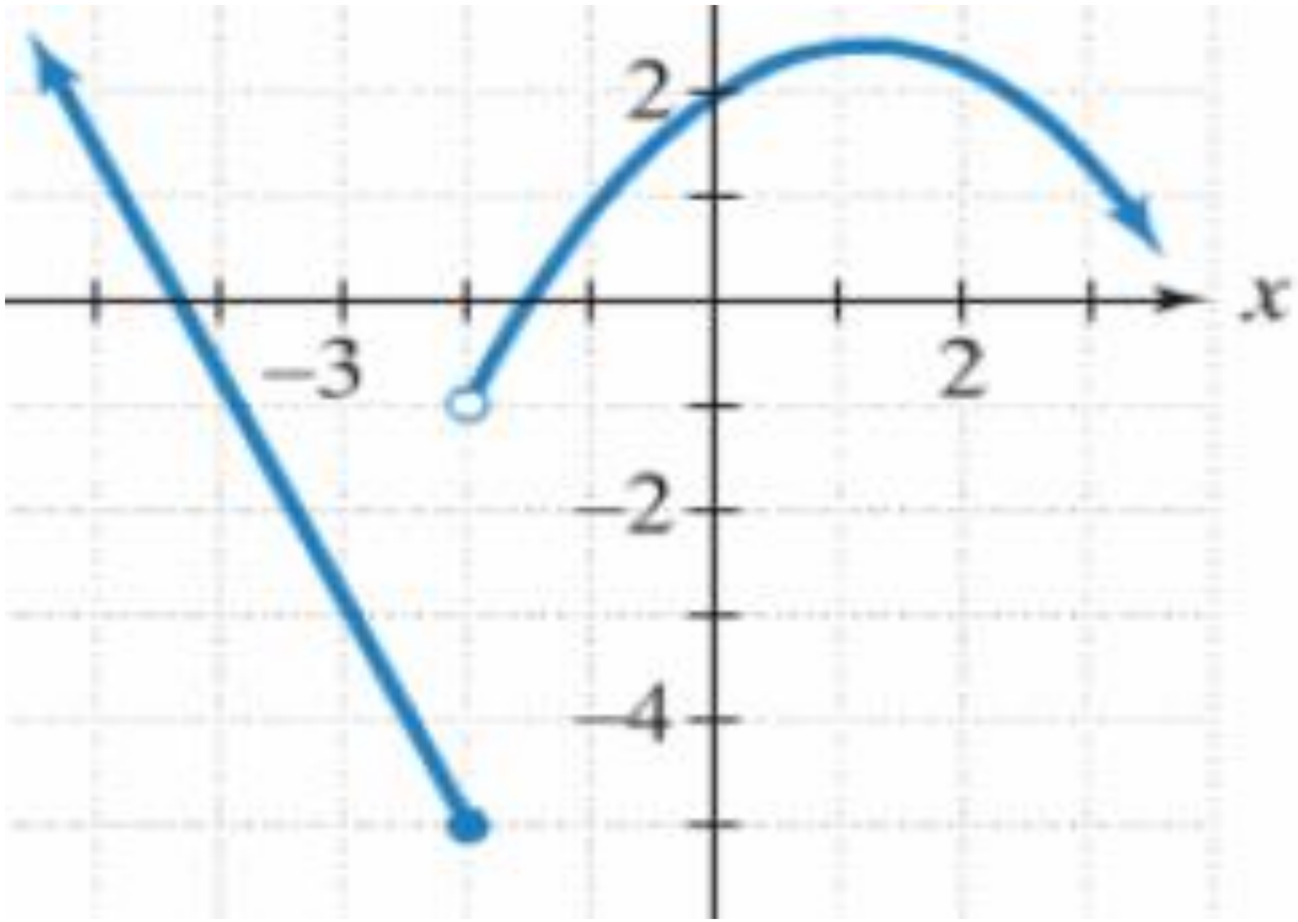
2. **Right-Hand Limits:** values of $x$ are larger than some number $a$

$$\lim_{x \to a^+} f(x) = M$$

# Visual Example

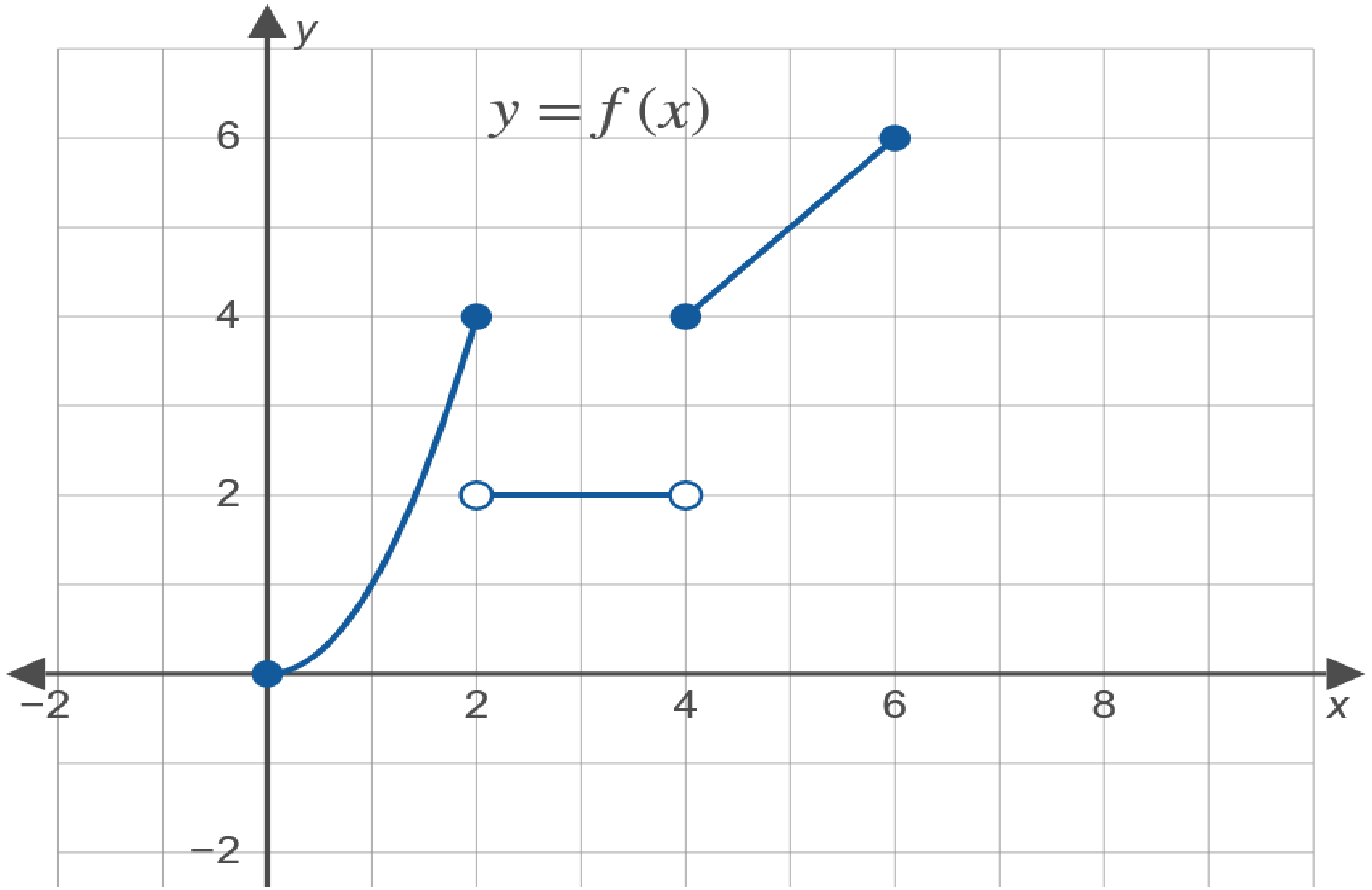
$$\lim_{x \to -2^-} f(x) =$$

$$\lim_{x \to -2^+} f(x) =$$



# Example: Limits from a graph

The graph of $f(x) = \begin{cases} x^2, & 0 \le x \le 2 \\ 2, & 2 < x < 4 \\ x, & 4 \le x \le 6 \end{cases}$ is shown. Find the following:




a. $\lim_{x \to 2^-} f(x)$

b. $\lim_{x \to 2^+} f(x)$

c. $\lim_{x \to 4^+} f(x)$

d. $\lim_{x \to 0^+} f(x)$

# Limits of Polynomial Functions

To find the limit of a polynomial function, we can use the method of "direct substitution".

If $p$ is a polynomial function and $a$ is a real number, then

$\lim_{x \to a^-} p(x) = p(a)$ and $\lim_{x \to a^+} p(x) = p(a)$

Example: $\lim_{x \to 3^-} (x^2 + 3x - 7)$

# More Examples

Find the following limits:

$$\lim_{x \to 2^+} (x^3 - 5x)$$

$$\lim_{x \to 0^-} (4x^2 - 8x + 3)$$

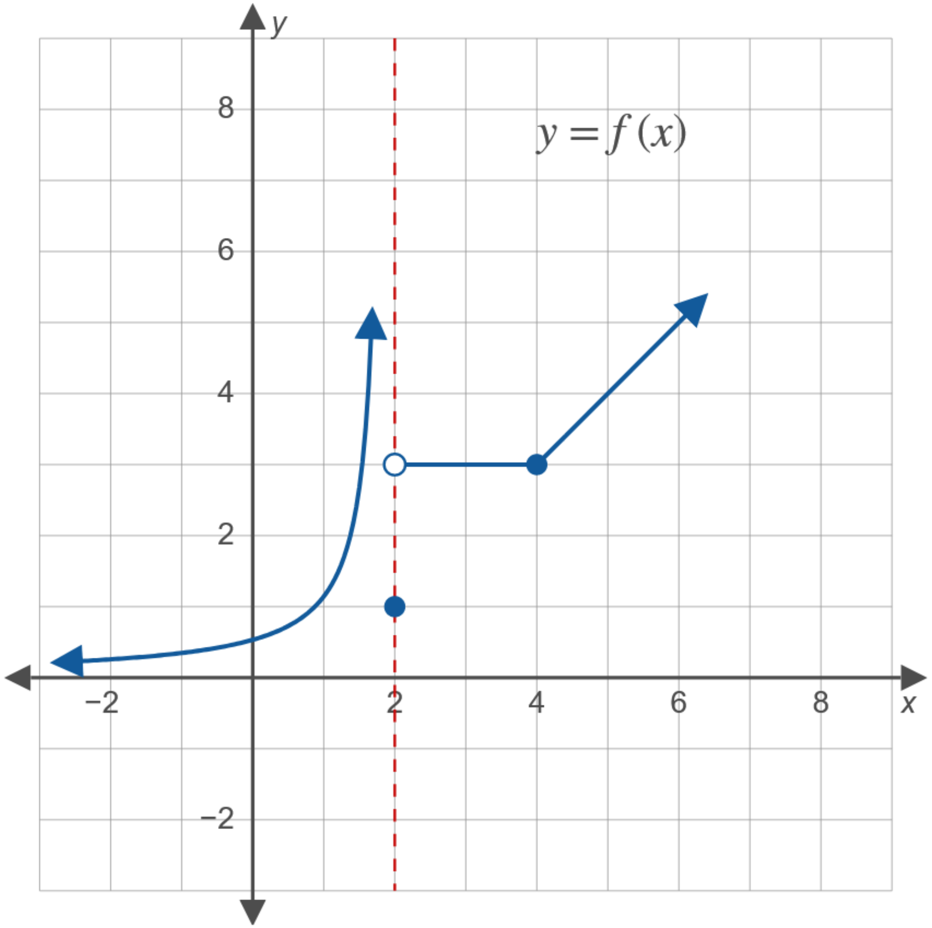
# Infinite Limits

1. If $f(x)$ increases without bound as $x$ approaches $a$ from the left (or from the right), then we say that $f(x)$ approaches positive infinity, $+\infty$. We write $\lim_{x \to a^-} f(x) = +\infty$ or $\lim_{x \to a^+} f(x) = +\infty$.

2. If $f(x)$ decreases without bound as $x$ approaches $a$ from the left (or from the right), then we say that $f(x)$ approaches negative infinity, $-\infty$. We write $\lim_{x \to a^-} f(x) = -\infty$ or $\lim_{x \to a^+} f(x) = -\infty$.

# Example:

Find the following limits:




a) $\lim_{x \to 2^-} f(x)$

b) $\lim_{x \to 2^+} f(x)$

c) $\lim_{x \to 4^-} f(x)$

d) $\lim_{x \to 4^+} f(x)$

# Analytical Example:

Find the limits:

$a)\ \lim_{x \to -3^-} \left(\frac{x+5}{x+3}\right)$

b) Find $\lim_{x \to -3^+} \left(\frac{x+5}{x+3}\right)$

# 2.2 Limits

# Existence of a Limit

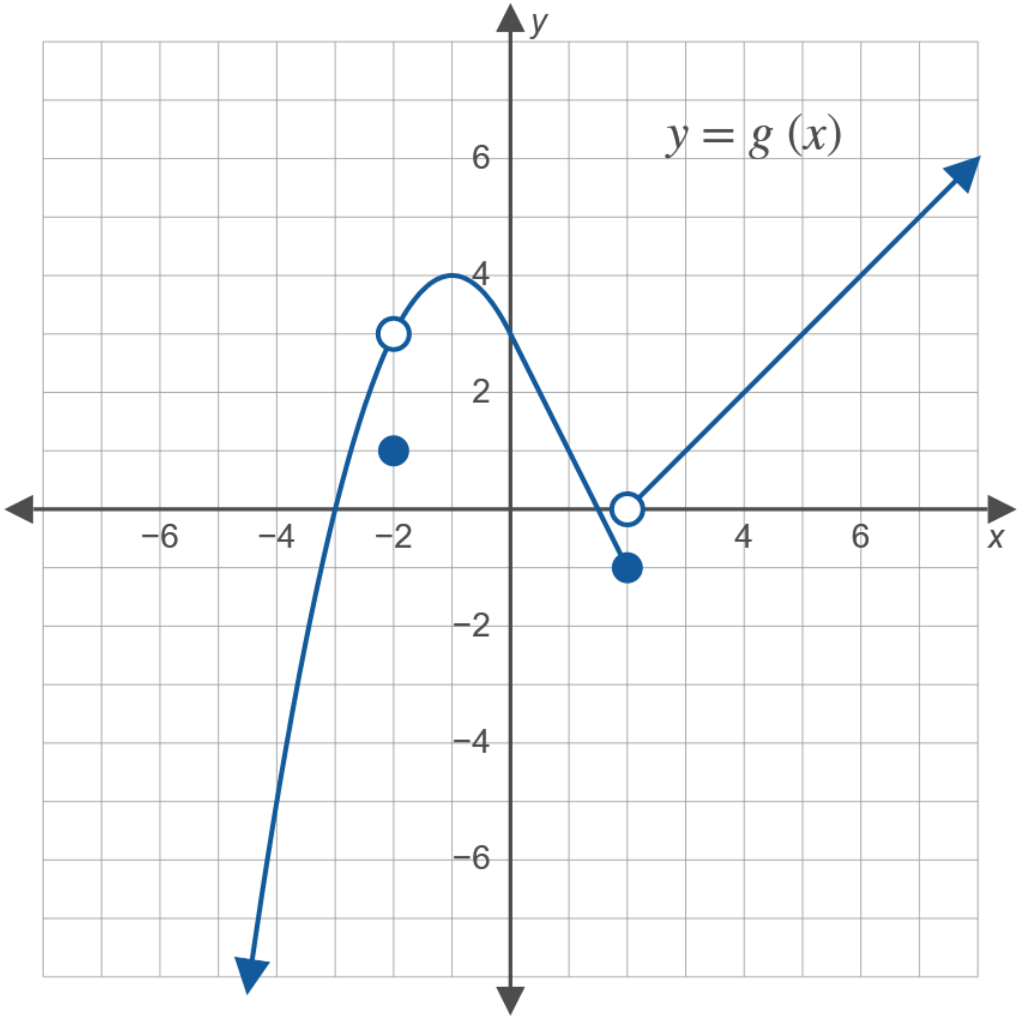
A limit exists if and only if:

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

Use the graph of $y = g(x)$ in the figure to find

a. $\lim_{x \to -2} g(x)$,

b. $\lim_{x \to 2} g(x)$,



# Using Direct Substition

1. $\lim_{x \to 3} (-6)$

2. $\lim_{x \to 7} 4x - 3$

3. $\lim_{x \to 2} (2x^2 + 3x \ - 1)^2$

# Using Simplification Techniques

1. $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$

2. $\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9}$

# Homework Examples

Determine $\lim_{x \to 2} \left( \frac{10(x^2 - 4)}{x - 2} \right)$ by using an algebraically equivalent expression.

Determine the $\lim_{x \to -3} \left( \frac{2x^2 + 5x - 3}{x + 3} \right)$

# Homework Examples

Given $f(x) = x^2 + 6$, find

$$\lim_{h \to 0} \left( \frac{f(7+h) - f(7)}{h} \right)$$