2.7 Definition of Derivative

Recall the Difference Quotient:

If $(x,\ f(x))$ is any fixed point on a curve and $(x+h,\ f(x+h))$ is another point on the curve, then the **difference quotient** is the slope of the secant line through these two points.

$$m_{\text{sec}} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

The Limit Process

By making the "h" smaller, we can make our secant closer to the tangent line. As h approaches 0, our secant slope approaches the tangent slope!

$$m_{\tan} = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$

Example

Find the slope of the graph of $f(x) = x^2 + 1$ at the point (-1, 2) and at the point (2, 5).

Generalizing the Limit Process

Find an equation for the slope of the graph of $f(x) = x^2 + 1$.

Definition of Derivative

For any given function $y = f(x)$,the derivative of f at x is defined to be

$$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$

provided this limit exists.

The derivative is the slope of the tangent line to f(x) at the point (x, y).

Derivative notations

- $f'(x)$
- $\frac{dy}{dx}$
- y'
- $\frac{d}{dx}[f(x)]$
- $D_x[y]$

Additional Vocabulary:

- The process of finding the derivative is “Differentiation”

- If you can take the derivative of a function, the function is “Differentiable”

Example 1

Find the derivative of the function:
$f(x) = x^2 – x + 6$

You try!

Find the derivative of the function:
$f(x) = 3x^2 + 4$

Finding the **equation** of Tangent line

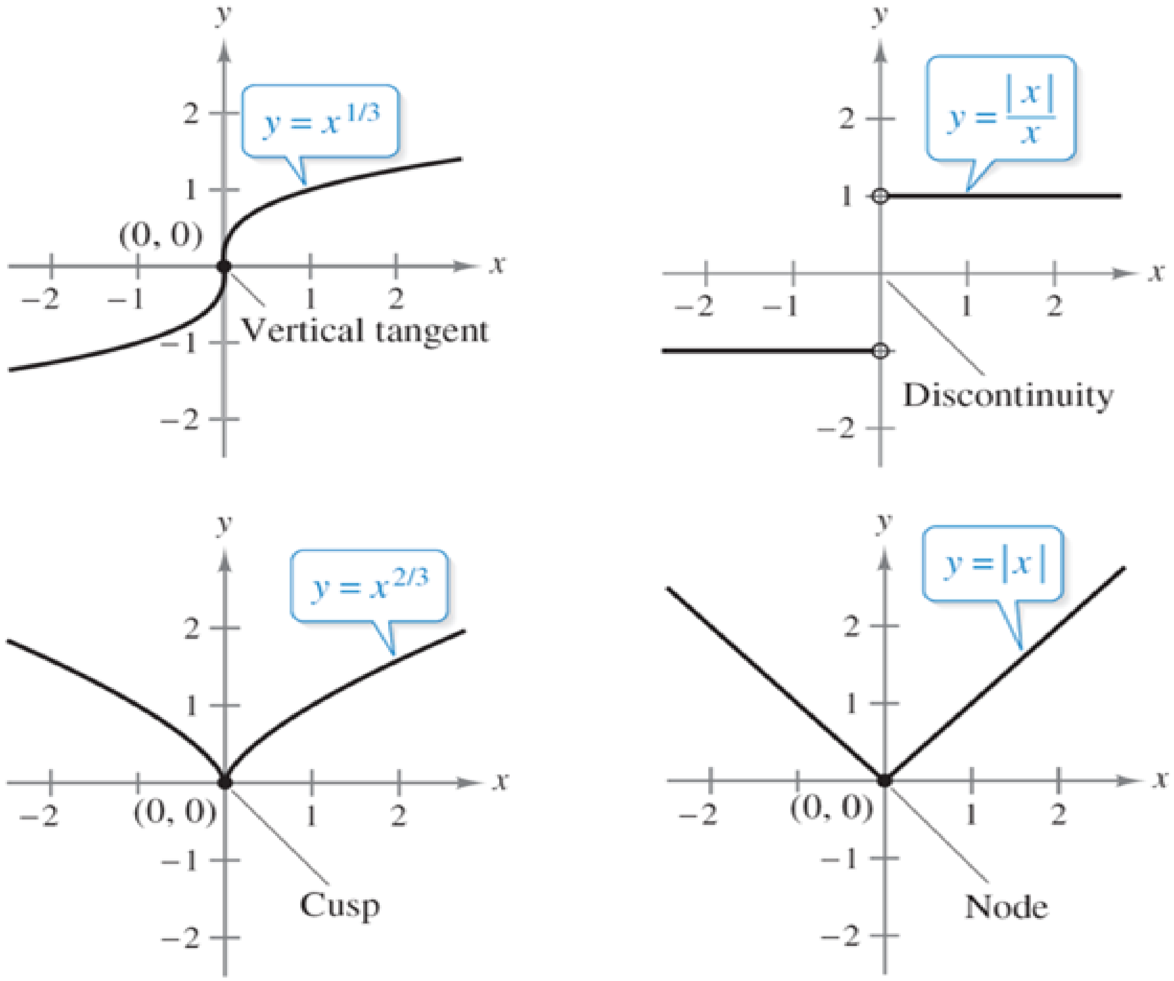
- To find the equation of a line, recall that we need a point and slope
- Point will be (c, f(c))
- Slope will be f '(c)
- Use y=mx + b or $y - y_1 = m(x - x_1)$

Example: Find the equation of the tangent line to $f(x) = x^2$ at the point (2, 4)

Differentiability and Continuity

Functions are not differentiable at:

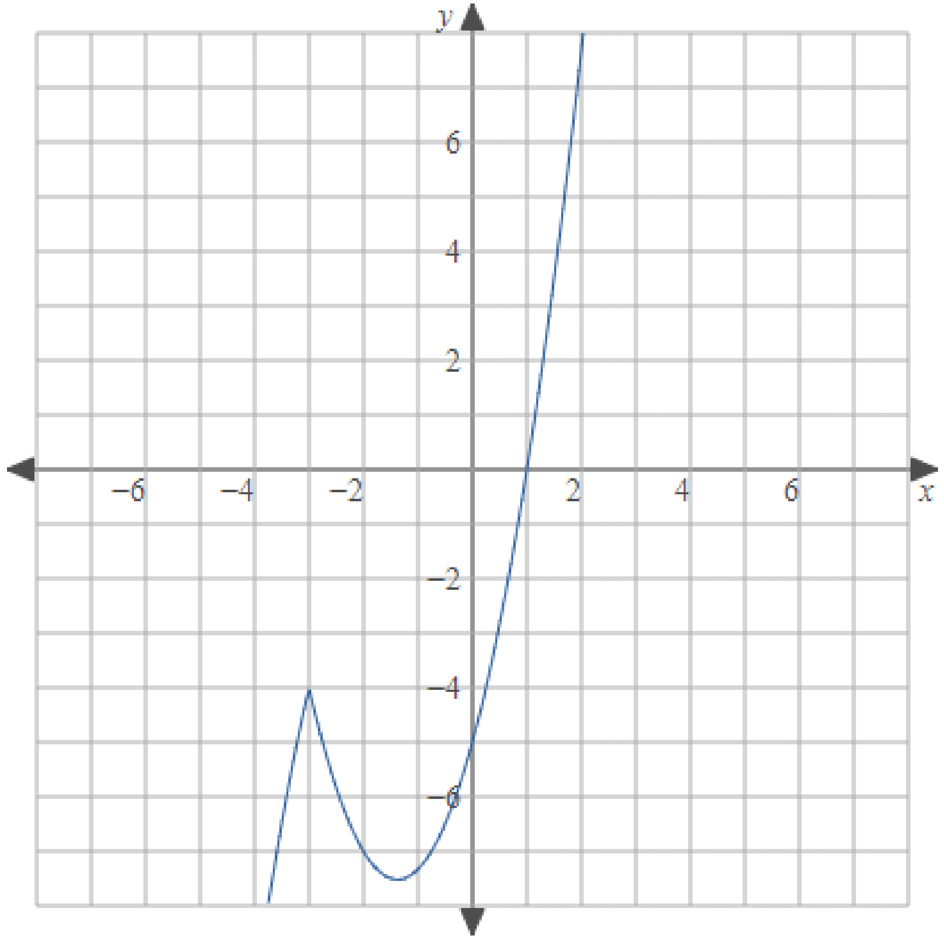
- Sharp points (Ex. Absolute value) (ve
- Discontinuities (Holes, vertical asymptotes)
- Vertical Tangent Lines



Functions That Are Not Differentiable at $x = 0$

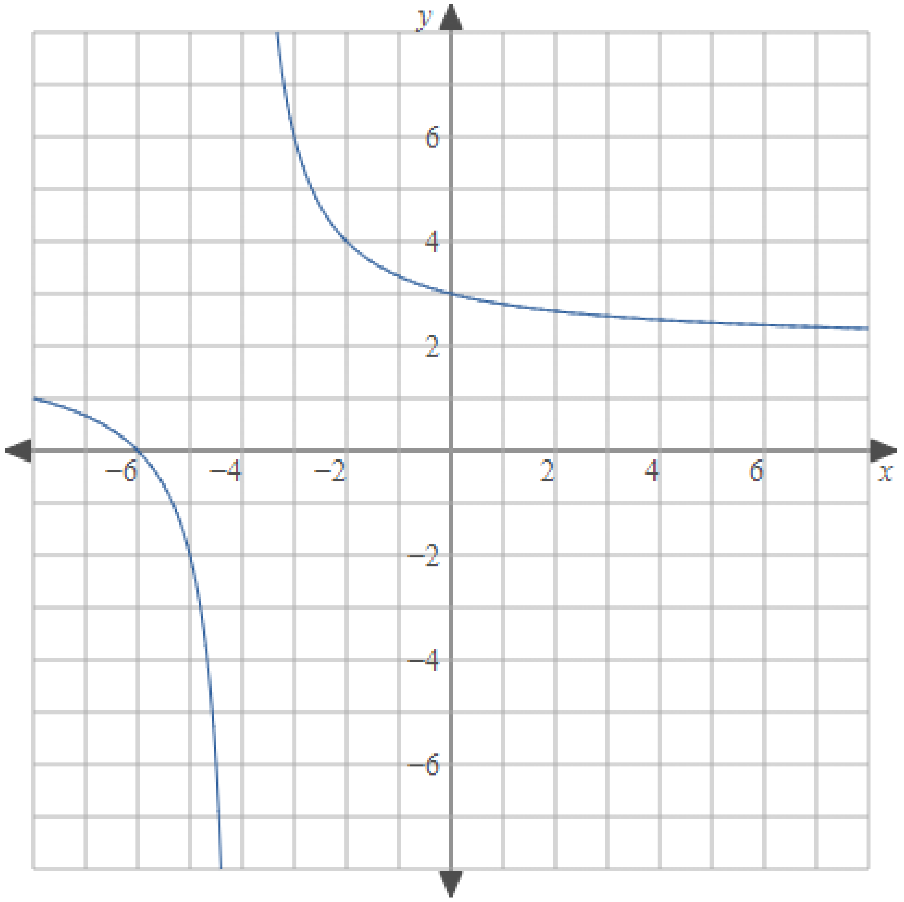
Differentiability

Find all values of x where the function with the following graph is not differentiable. Separate multiple answers with a comma. Enter *None* if the function is differentiable everywhere.



Differentiability

Find all values of x where the function with the following graph is not differentiable. Separate multiple answers with a comma. Enter *None* if the function is differentiable everywhere.



Differentiability and Continuity - continued

If a function is differentiable at *x=c,* then is continuous at *x=c.*

However, if a function is continuous, we don't know for sure if it is differentiable. An example is $f(x) = |x|$. *f(x)* is continuous at *x=0* but not differentiable at *x=0* (sharp point).

2.8 Basic Derivative Rules

Given the following, identify a pattern:

$f(x) = x^2$
$f'(x) = 2x$

$f(x) = x^3$
$f'(x) = 3x^2$

$f(x) = x^4$
$f'(x) = 4x^3$

Basic Derivatives

Derivative of a constant:

f (x) = c

f '(x) = 0

Power rule:

f(x) = x^n

f '(x) = (n)(x^{n-1})

Constant Multiple:

If f is a differentiable of x, and c is a real number, then

$$\frac{d}{dx}[cf(x)] = cf'(x)$$

Where c is a constant.

Note that:

$$\frac{d}{dx}[cx] = c$$

Examples: Find the derivative

$$f(x) = x$$

$$f(x) = x^{10}$$

$$y = 5x^3$$

$$g(x) = x^2$$

$$y = -\frac{3x}{2}$$

$$y = x^3$$

Examples: Find the Derivative by rewriting first

1. $g(x) = \sqrt[3]{x}$

3. $\mathrm{y} = \frac{1}{\sqrt[3]{x^2}}$

2. $y = \frac{1}{x^3}$

4. $y = (3x)^2$

Theorem 2.5 **Sum and Difference Rule**

The sum (or difference) of two differentiable functions f and g is itself differentiable. Moreover, the derivative of *f+g* (or *f-g*) is the sum (or difference) of the derivatives of f and g.

$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$$

Examples: Find the derivative

$$y = x^3 + 5x^2 - 3x + 8$$

$$f(x) = \sqrt{x} - 6\sqrt[3]{x}$$

$$f(x) = \frac{6x^2 - 3x + 7}{x}$$

Example: Find the Equation of the Tangent line

$$g(x) = -\frac{1}{2}x^4 + 3x^3 - 2x \quad at\ the\ point\ \left(-1, -\frac{3}{2}\right)$$

Application: Velocity

- Suppose that a sailboat is observed, over a period of 5 minutes, to travel a distance from a starting point according to the function $s(t) = t^3 + 60t$,where t is time in minutes and s is the distance traveled in meters.

a. How fast is the boat moving at the starting point?

b. How fast is the boat moving at the end of 3 minutes?

2.9 Marginal Analysis

Business Terminology

Profit = Revenue – Cost $P(x) = R(x) - C(x)$

Revenue = (demand)(units) $\mathrm{R}(x) = p * x$

Cost = (cost per unit)(units) + fixed cost $C(x) = cx + k$

Average cost = Cost / units $\bar{C}(x) = \frac{C(x)}{x}$

Marginal Cost, Revenue, Profit

Marginal Cost = C'(x)

Marginal Revenue = R'(x)

Marginal Profit = P'(x) = R'(x) – C'(x)

Marginal Average Cost = $\bar{C}'(x) = \frac{d}{dx}\left[\frac{C(x)}{x}\right]$

Marginal Cost

A manufacturer of specialty items determines that the cost of producing x ballpoint pens is $C(x) = 500 + 3x$ in dollars.

a. Find $C(101) - C(100)$,the cost of producing the 101st pen.
b. Find $C'(100)$,the marginal cost at $x = 100$ pens.

Marginal Revenue

Suppose that a manufacturer has determined that the price of certain custom-made tables he produces can be determined by the demand function $p = D(x) = 117 - \frac{x}{4}$,where x is the number of tables produced and sold.

a. Find the revenue function.
b. Determine the marginal revenue when $x = 16$ tables.

Marginal Profit

If the table manufacturer in Example 2 has a cost function of $C(x) = 225 + 2x^2$ along with his revenue function of

$R(x) = 117x - \frac{x^2}{4}$,

find

a. all break-even points;
b. the marginal profit when $x = 10, x = 20, x = 30$.

Marginal Average Cost

Suppose $C(x) = 0.02x^2 + 100x + 2000$ represents the cost in dollars of producing x lawn mowers.

a. Find the average cost function.
b. Find the average cost per lawn mower if 1000 lawn mowers are produced.
c. Find the marginal average cost if 1000 lawn mowers are produced